# MEC ENG 103: Probability, Statistics, and Signals Reference Sheet

## Probabilities for Events

Komolgorov's Axioms of Probabilities:
1. $\mathbb{P}(A) \geq 0 \quad \forall A \subseteq \Omega$
2. $\mathbb{P}(\Omega) = 1$
3. $\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$

For events $A, B,$ and $C$ : $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
More generally, we have the inclusion-exclusion principle:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots$$

The odds in favour of $A$: $\mathbb{P}(A)/\mathbb{P}(\overline{A})$ where $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$
Conditional probability: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ provided $\mathbb{P}(B) > 0$.
Chain Rule or Product Rule:

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$$

Bayes' Rule:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A)\mathbb{P}(B|A)}{\mathbb{P}(A)\mathbb{P}(B|A) + \mathbb{P}(\overline{A})\mathbb{P}(B|\overline{A})}$$

Law of Total Probability: $\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$
$A$ and $B$ are independent $\iff$ $\mathbb{P}(B|A) = \mathbb{P}(B)$
$A$, $B$, and $C$ are independent $\iff$ $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$,
$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$, $\mathbb{P}(C \cap A) = \mathbb{P}(C)\mathbb{P}(A)$

## Distributions, Expectation, and Variance

The probability distribution for a discrete random variable $X$ is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{\mathbb{P}(X = x)\}$
The expected Value is $\mathbb{E}[X] = \mu = \sum_x x p_x$
For function $g(x)$ of $x$, $\mathbb{E}[g(X)] = \sum_x g(x)p_x$, so $\mathbb{E}[X^2] = \sum_x x^2 p_x$
Sample mean $\overline{x} = \frac{1}{n}\sum_k x_k$ estimates $\mu$ from sample $x_1, \dots, x_n$
Variance $\mathrm{Var}[X] = \sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mu^2$
Sample Variance $s^2 = \frac{1}{n-1}\left(\sum_j^n x_j^2 - \frac{1}{n}\left(\sum_j^n x_j\right)^2\right)$ estimates $\sigma^2$
Standard Deviation $\mathrm{sd}(X) = \sigma$
If value $y$ is observed with frequency $n_y$

$$n = \sum_y n_y,\ \sum_k x_k = \sum_y y n_y,\ \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\beta_1 = \mathbb{E}\left(\frac{X-\mu}{\sigma}\right)^3$ is estimated by $\frac{1}{n-1}\sum_i \left(\frac{x_i - \bar{x}}{s}\right)^3$
Kurtosis $\beta_2 = \mathbb{E}\left(\frac{X-\mu}{\sigma}\right)^4 - 3$ is estimated by $\frac{1}{n-1}\sum_i \left(\frac{x_i - \bar{x}}{s}\right)^4 - 3$
Sample median $\tilde{x}$ or $x_{med}$. Half the sample values are smaller and half larger. If the sample values $x_1, \dots, x_n$ are ordered as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$, then $\tilde{x} = x_{((n+1)/2)}$ if $n$ is odd, and $\tilde{x} = \frac{1}{2}(x_{(n/2)} + x_{((n+2)/2)})$
$\alpha$-quantile $Q(\alpha)$ is such that $\mathbb{P}(X \leq Q(\alpha)) = \alpha$
Sample $\alpha$-quantile $\hat{Q}(\alpha)$ proportion $\alpha$ of the data values are smaller
Lower Quartile $Q1 = \hat{Q}(0.25)$ one quarter are smaller
Upper Quartile $Q3 = \hat{Q}(0.75)$ three quarters are smaller
Sample median $\tilde{x} = \hat{Q}(0.5)$ estimates the pop. median $Q(0.5)$

## Discrete Probability Distributions

Discrete Uniform: $X \sim \mathrm{Uniform}(n)$

$$p_x = \frac{1}{n}\ (x = 1, 2, \dots, n) \quad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$$

Binomial Distribution: $X \sim \mathrm{Binomial}(n, \theta)$

$$p_x = \binom{n}{x}\theta^x(1-\theta)^{n-x}\ (x = 1, 2, \dots, n) \quad \mu = n\theta,\ \sigma^2 = n\theta(1-\theta)$$

Poisson Distribution: $X \sim \mathrm{Poisson}(\lambda)$

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!}\ (x = 1, 2, \dots)\ (\text{with } \lambda > 0) \quad \mu = \lambda,\ \sigma^2 = \lambda$$

Geometric Distribution: $X \sim \mathrm{Geometric}(\theta)$

$$p_x = (1-\theta)^{x-1}\theta\ (x = 1, 2, \dots) \quad \mu = \frac{1}{\theta},\ \sigma^2 = \frac{1-\theta}{\theta^2}$$

## Continuous Random Variables

The cum. distr. fn. (cdf): $F(x) = \mathbb{P}(X \leq x) = \int_{x_0=\infty}^{x} f(x_0)dx_0$
The probability density fn. (pdf): $f(x) = dF(x)/dx$

$$\mathbb{E}[X] = \mu = \int_{-\infty}^{\infty} x f(x)dx,\ \mathrm{Var}[X] = \sigma^2 = \mathbb{E}[X^2] - \mu^2$$

where $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$

## Permutations and Combinations

Permutation with **repetition**: $n^r$
Permutation with **no repetition**: ${}_nP_r = \frac{n!}{(n-r)!}$
Combination with **repetition**: $\frac{(n+r-1)!}{r!(n-1)!}$
Combination with **no repetition**: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
Counting ways to partition $n$ objects into $n^i$ groups

$$\binom{n}{n_1, n_2 \dots n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$$

## Bounds

Markov Inequality: $\mathbb{P}[X \geq a] \leq \mathbb{E}[X]/a$
Chebyshev's Inequality: $\mathbb{P}(|X - \mu| \geq c) \leq \sigma^2/c^2$
Chernoff Bound: $\mathbb{P}[X \geq a] \leq \mathbb{E}[e^{sx}]/e^{sa}, s > 0$
Jensen's Inequality: $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$, $f$ is convex, $f''(x) > 0$
Union Bound

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

Weak Law of Large Numbers is as follows

$$\lim_{n\to\infty} \mathbb{P}\left[|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X]| \geq \epsilon\right] = 0$$

Strong Law of Large Numbers: $\mathbb{P}[\lim_{n\to\infty} M_n = \mu] = 1$

## Continuous Probability Distributions

Uniform distribution: $X \sim \mathrm{Uniform}(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & \text{otherwise} \end{cases}, \quad \mu = \frac{\alpha+\beta}{2},\ \sigma^2 = \frac{(\beta-\alpha)^2}{12}$$

Exponential distribution: $X \sim \mathrm{Exponential}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0) \end{cases}, \quad \mu = 1/\lambda,\ \sigma^2 = 1/\lambda^2$$

Normal distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],\ (-\infty < x < \infty)$$

with the following parameters $\mathbb{E}[X] = \mu$ and $\mathrm{Var}[X] = \sigma^2$
Standard Normal distribution: $X \sim \mathcal{N}(0, 1)$

## Reliability

For a device in continuous operation with failure time random variable $T$ having a pdf $f(t)$ given $t > 0$, then:
The reliability function at time $t$: $R(t) = \mathbb{P}(T > t)$
The failure rate or hazard function: $h(t) = f(t)/R(t)$
The cumulative hazard function: $H(t) = \int_0^t h(t_0)dt_0 = -\ln R(t)$
The Weibull$(\alpha, \beta)$ distribution has $H(t) = \beta t^{\alpha}$
The mean time to failure: $MTTF = \mathbb{E}[T] = -\int_0^{\infty} tR'(t)dt$

## System Reliability

For a system of $k$ devices, which operate *independently*, let $R_i = \mathbb{P}(D_i) = \mathbb{P}$("device $i$ operates")
The system reliability, $R$, is the probability of a path of operating devices
A system of devices in series operates only if every device operates

$$R = \mathbb{P}(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = \mathbb{P}(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

# MEC ENG 103: Probability, Statistics, and Signals Reference Sheet

## Covariance and Correlation

The covariance of $X$ and $Y$: $\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ we have the following relationships

$$S_{xy} = \sum_k x_k y_k - \frac{1}{n} \sum_i x_i \sum_j y_j$$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2,\ S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample Covariance: $s_{xy} = S_{xy}/(n-1)$ estimates $\text{Cov}(X,Y)$
Correlation Coeff.: $\rho = \text{Corr}(X,Y) = \text{Cov}(X,Y)/(\text{sd}(X)\text{sd}(Y))$
Sample correlation coeff.: $r = S_{xy}/\sqrt{S_{xx}S_{yy}}$ estimates $\rho$

## Sums of Random Variables

$\mathbb{E}[\cdot]$ is a linear operator: $E[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X,Y]$
For 2 random variables $J = \alpha X + \beta Y$ and $Z = \gamma X + \delta Y$
$\text{Cov}[J,Z] = (\alpha\gamma)\text{Var}[X] + (\beta\delta)\text{Var}[Y] + (\alpha\delta + \beta\gamma)\text{Cov}[X,Y]$
If $X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and $\text{Cov}[X,Y] = c$, then adding the RVs yields $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2c)$

## Bias, Standard Error, and MSE

If $t$ estimates $\theta$ (with random variable $T$ giving $t$) we have Bias of $t$: $\text{bias}(t) = \mathbb{E}[T] - \theta$
Standard error of $t$: $\text{se}(t) = \text{sd}(T)$
MSE of $t$: $\text{MSE}(t) = \mathbb{E}[(T-\theta)^2] = (\text{se}(t))^2 + (\text{bias}(t))^2$
If $\bar{x}$ estimates $\mu$, then $\text{bias}(\bar{x}) = 0$, $\text{se}(\bar{x}) = \sigma/\sqrt{n}$, $\text{MSE}(\bar{x}) = \sigma^2.n$, $\widehat{\text{se}}(\bar{x}) = s/\sqrt{n}$
Central Limit Theorem: If $n$ is fairly large i.e. $n \geq 30$, $\bar{x}$ is from $\mathcal{N}(\mu, \sigma^2/n)$ approximately.

## Likelihood

If we observe $\{Y = y\}$, then the a posteriori probability fo $\{X = x\}$ is given by

$$\mathbb{P}(X = x|Y = y) = \frac{p_{Y|X}(y|x)\pi(x)}{\sum_{\tilde{x}} p_{Y|X}(y|\tilde{x})\pi(\tilde{x})} \propto p_{Y|X}(y|x)\pi(x)$$

Our prior has been updated given observations $Y$. This motives the maximum a posteriori (MAP).

$$\hat{X}_{\text{MAP}}(y) = \arg\max_x p_{Y|X}(y|x)\pi(x) = \arg\max p_{X|Y}(x|y)$$

The likelihood is the joint probability as a function of the unknown parameter $\theta$. For a random sampe $x_1, x_2, \ldots, x_n$

$$\ell(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i|\theta) \quad \text{(discrete distribution)}$$

$$\ell(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p(x_i|\theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\hat{\theta}$ for which the likelihood is a maximum.

## Confidence Intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $\mathcal{N}(\mu, \sigma^2)$ and $\sigma^2$ is known, then the 95% confidence interval for $\mu$ is $(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}})$. If $\sigma^2$ is estimated, then from the $t$-table for $t_{n-1}$ we find $t_0 = t_{n-1,0.05}$. The 95% confidence interval for $\mu$ is $(\bar{x} - t_0\frac{s}{\sqrt{n}}, \bar{x} + t_0\frac{s}{\sqrt{n}})$
Central Limit Theorem: $\lim \mathbb{P}(\widehat{S_n} \leq x) = \Phi(x)$

## The Chi-Squared GOF Test

To test the goodness-of-fit of a probability model to a sample of size $n$, use the Chi-squared statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

If $H_0$ is true, then $\chi^2$ approximately has a Chi-squared distribution with $k - d - 1$ degrees of freedom, where $d$ is the number of estimated parameters.

## Hypothesis Testing

One-sample test for the mean, $z$-test and $t$-test

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}},\ t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Two-sample test for the mean (equal variances) and $\mu_1 - \mu_2 = 0$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}},\ t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where we define the pooled standard deviation as

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Two-sample test of the mean (unequal variances) with $\mu_1 - \mu_2 = 0$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{SE},\ SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

The DoF for above is given by the Welch-Satterwaite Equation:

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$

## Joint Probability Distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = \mathbb{P}(\{X = x\} \cap \{Y = y\})$
The marginal distributions are given as

$$p_X(x) = \mathbb{P}(X = x) = \sum_y p_{xy},\ p_Y(y) = \mathbb{P}(Y = y) = \sum_x p_{xy}$$

Then the conditional distribution of $X$ given $Y = y$ is

$$\mathbb{P}(X = x|Y = y) = p_{xy}/p_Y(y)$$

## Linear Regression

To fit the linear regression model $y = \alpha + \beta x$ by $\hat{y}_x = \hat{\alpha} + \hat{\beta}x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\hat{\alpha} = \bar{y} - \bar{x}\hat{\beta}$, $\hat{\beta} = S_{xy}/S_{xx}$.
The residual sum of squares: $RSS = S_{yy} - S_{xy}^2/S_{xx}$ with $\widehat{\sigma^2} = RSS/(n-2)$ where $RSS$ is from $\chi^2_{n-2}$.

## Wilcoxon Rank Sum Test

One-tailed test: Test statistic: $T_1$, if $n_1 < n_2$; $T_2$, if $n_2 < n_1$; either rank sum can be used if $n_1 = n_2$
**Rejection Region:** $T_1 \geq T_U$ (or $T_1 \leq T_L$); $T_2 \leq T_U$ (or $T_2 \geq T_U$), where $T_L$ and $T_U$ are obtained from the tables on the next page.

Two-tailed test: Test statistic: $T_1$, if $n_1 < n_2$; $T_2$, if $n_2 < n_1$; either rank sum can be used if $n_1 = n_2$. We will denote this rank sum as $T$.
**Rejection Region:** $T \leq T_L$ (or $T \geq T_U$), where $T_L$ and $T_U$ are obtained from the table on the next page.

## Wilcoxon Rank Sum Test ($n_1 \geq 10$ and $n_2 \geq 10$)

One-tailed test: the test statistic is given by

$$Z_c = \frac{T_1 - 0.5(n_1 n_2 + n_1(n_1 + 1))}{\sqrt{1/12(n_1 n_2(n_1 + n_2 + 1))}}$$

**Rejection Region**: $Z_c > z_\alpha$ (or $Z_c < -z_\alpha$)

Two-tailed test: the test statistic is given by

$$Z_c = \frac{T_1 - 0.5(n_1 n_2 + n_1(n_1 + 1))}{\sqrt{1/12(n_1 n_2(n_1 + n_2 + 1))}}$$

**Rejection Region**: $|Z_c| > z_{\alpha/2}$

# MEC ENG 103: Probability, Statistics, and Signals Reference Sheet

## Wilcoxon Signed Ranks Test

One-tailed test: Test statistic: $T_-$ is the negative rank sum (or $T_+$ is the positive rank sum)
**Rejection Region**: $T_- \leq T_0$ (or $T_+ \leq T_0$), where $T_0$ is obtained from the tables on the next page.

Two-tailed test: Test statistic: $T$, the smallest of $T_-$ which is the negative rank sum or $T_+$ is the positive rank sum.
**Rejection Region**: $T \leq T_0$, where $T_0$ is obtained from the table on the next page.

## Wilcoxon Signed Ranks Test $n \geq 25$

One-tailed test: the test statistic is given by

$$Z_c = \frac{T_+ - (n(n+1)/4)}{\sqrt{(n(n+1)(2n+1))}}$$

**Rejection Region**: $Z_c > z_\alpha$ (or $Z_c < -z_\alpha$)

Two-tailed test: the test statistic is given by

$$Z_c = \frac{T_+ - (n(n+1)/4)}{\sqrt{(n(n+1)(2n+1))}}$$

**Rejection Region**: $|Z_c| > z_{\alpha/2}$

## Kruskal-Wallis $H$ Test

Non-parametric alternative to one-way ANOVA. Use when you have **3 or more independent** groups. Test Statistic given by:

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{T_i^2}{n_i} - 3(n+1)$$

where $n_i$ is the number of measurements in sample $i$; $T_i$ is the rank sum for sample $i$, where the rank of each measurement is computed according to its relative magnitude in the totality of data for the $k$ samples; $n$ is the total sample size i.e. $n = n_1 + n_2 + \cdots + n_k$.
**Rejection Region**: $H > \chi^2_\alpha$ with $k-1$ degrees of freedom.

## Friedman Test

Non-parametric alternative to repeated measures ANOVA. Use when you have **3 or more related**, paired, or repeated measurements from the same group of subjects **and** you want to see if there is a difference between these conditions or time points. Test Statistic given by:

$$F_r = \frac{12}{bk(k+1)} \sum_{i=1}^{k} T_i^2 - 3b(k+1)$$

where $b$ is the # of blocks employed in the experiment; $k$ is the # of treatments; $T_i$ is the sum of the ranks for the $i$th treatment.
**Rejection Region**: $F_r > \chi^2_\alpha$ with $k-1$ degrees of freedom.

## CT and DT Fourier Series

DTFS and CTFS Analysis Equations

$$X_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-ik\omega_0 n}, \quad X_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-ik\omega_0 t} dt$$

DTFS and CTFS Synthesis Equations

$$x[n] = \sum_{k \in \langle N \rangle} X_k e^{ik\omega_0 n}, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

**Note**: $\langle p \rangle$ represents a period of $p$ (e.g. $(-\frac{1}{2}p,\ \frac{1}{2}p)$ or $(0,\ p)$). For the equations, $\omega \leftrightarrow 2\pi f$, $\omega_0 \leftrightarrow 2\pi f_0$, and $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ or $N = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

## Discrete Fourier Transform Equations

DFT Analysis and Synthesis Equations

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-ik\omega_0 n}, \quad x[n] = \sum_{k=0}^{N-1} X[k] e^{ik\omega_0 n}$$

**Note**: $\langle p \rangle$ represents a period of $p$ (e.g. $(-\frac{1}{2}p,\ \frac{1}{2}p)$ or $(0,\ p)$). For the equations, $\omega \leftrightarrow 2\pi f$, $\omega_0 \leftrightarrow 2\pi f_0$, and $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ or $N = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

## DTFT and CTFT Equations

DTFT and CTFT Analysis Equations

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega n}, \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

DTFT and CTFT Synthesis Equations

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{i\omega n} d\omega, \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega$$

**Note**: $\langle p \rangle$ represents a period of $p$ (e.g. $(-\frac{1}{2}p,\ \frac{1}{2}p)$ or $(0,\ p)$). For the equations, $\omega \leftrightarrow 2\pi f$, $\omega_0 \leftrightarrow 2\pi f_0$, and $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ or $N = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

## Uncertainty Propagation

For a given function $Y = f(x_1, x_2, \ldots, x_n)$, uncertainty $u_i$ in each $x_i$ propagates to the uncertainty of $Y$

$$u_Y = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} u_i \right)^2}$$

## Impedance

**Impedance** is comprised of a resistance (R) a real component and reactance (X) an imaginary component

$$\mathbb{Z} = R + jX$$

**Capacitor Impedance**: $\mathbb{Z} = \frac{1}{j\omega C} = \frac{1}{j(2\pi f)C}$
**Inductor Impedance**: $\mathbb{Z} = j\omega L = j(2\pi f)L$

## Filters

Ideal Low-Pass Filter

$$\frac{v_o}{v_i} = \frac{1}{1 + j\omega RC}, \; f_c = \frac{1}{2\pi RC}$$

Ideal High-Pass Filter

$$\frac{v_0}{v_i} = \frac{1}{1 + j\omega \frac{L}{R}} = \frac{j\omega RC}{1 + j\omega RC}, \; f_c = \frac{R}{2\pi L}$$

Notch Filter

$$\frac{s^2 + \omega_0^2}{s^2 + Q^{-1}s + \omega_0^2}, \quad \text{where } s = j\omega$$

## Bode Plots

Magnitude Equation: $|H(j\omega)| = 20 \log_{10} |H(j\omega)|$
Phase Equation: $\angle H(j\omega) = \arctan(-\omega RC)$
For a stable nth order OL pole we have -20n dB/dec and at frequency $\omega$, from $0.1\omega$ to $10\omega$, phase changes linearly by -90n.
For a stable nth order OL zero we have +20n dB/dec and at frequency $\omega$, from $0.1\omega$ to $10\omega$, phase changes linearly by +90n.
If the effects of any p/z coincide with another, sum the effects up.

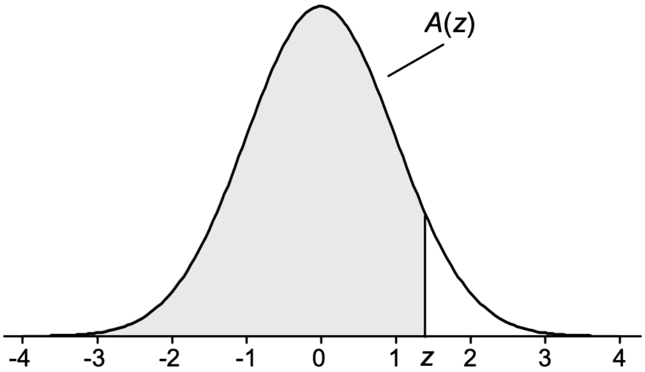
## Nyquist Sampling Theorem

Nyquist's Theorem states that to reconstruct a continuous analog signal $x(t)$ from its sampled version accurately, the sampling rate $f_s$ must be at least twice the highest frequency present in the signal $f_{max}$.

$$f_s \geq 2f_{max}$$

## 1 Cumulative Standardized Normal Distribution

$A(z)$ is the integral of the standardized normal distribution from $-\infty$ to $z$ (in other words, the area under the curve to the left of $z$). It gives the probability of a normal random variable not being more than $z$ standard deviations above its mean.




| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |
| 3.5 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| 3.6 | 0.9998 | 0.9998 | 0.9999 | | | | | | | |

## 2 $t$ Distribution: Critical Values of $t$

| Degrees of freedom | Two-tailed test: / One-tailed test: | 10% / 5% | 5% / 2.5% | 2% / 1% | 1% / 0.5% | 0.2% / 0.1% | 0.1% / 0.05% |
|---|---|---|---|---|---|---|---|
| | *Significance level* | | | | | | |
| **1** | | 6.314 | 12.706 | 31.821 | 63.657 | 318.309 | 636.619 |
| **2** | | 2.920 | 4.303 | 6.965 | 9.925 | 22.327 | 31.599 |
| **3** | | 2.353 | 3.182 | 4.541 | 5.841 | 10.215 | 12.924 |
| **4** | | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 | 8.610 |
| **5** | | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 | 6.869 |
| **6** | | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| **7** | | 1.894 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| **8** | | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| **9** | | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| **10** | | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| **11** | | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
| **12** | | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 | 4.318 |
| **13** | | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 | 4.221 |
| **14** | | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 | 4.140 |
| **15** | | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 | 4.073 |
| **16** | | 1.746 | 2.120 | 2.583 | 2.921 | 3.686 | 4.015 |
| **17** | | 1.740 | 2.110 | 2.567 | 2.898 | 3.646 | 3.965 |
| **18** | | 1.734 | 2.101 | 2.552 | 2.878 | 3.610 | 3.922 |
| **19** | | 1.729 | 2.093 | 2.539 | 2.861 | 3.579 | 3.883 |
| **20** | | 1.725 | 2.086 | 2.528 | 2.845 | 3.552 | 3.850 |
| **21** | | 1.721 | 2.080 | 2.518 | 2.831 | 3.527 | 3.819 |
| **22** | | 1.717 | 2.074 | 2.508 | 2.819 | 3.505 | 3.792 |
| **23** | | 1.714 | 2.069 | 2.500 | 2.807 | 3.485 | 3.768 |
| **24** | | 1.711 | 2.064 | 2.492 | 2.797 | 3.467 | 3.745 |
| **25** | | 1.708 | 2.060 | 2.485 | 2.787 | 3.450 | 3.725 |
| **26** | | 1.706 | 2.056 | 2.479 | 2.779 | 3.435 | 3.707 |
| **27** | | 1.703 | 2.052 | 2.473 | 2.771 | 3.421 | 3.690 |
| **28** | | 1.701 | 2.048 | 2.467 | 2.763 | 3.408 | 3.674 |
| **29** | | 1.699 | 2.045 | 2.462 | 2.756 | 3.396 | 3.659 |
| **30** | | 1.697 | 2.042 | 2.457 | 2.750 | 3.385 | 3.646 |
| **32** | | 1.694 | 2.037 | 2.449 | 2.738 | 3.365 | 3.622 |
| **34** | | 1.691 | 2.032 | 2.441 | 2.728 | 3.348 | 3.601 |
| **36** | | 1.688 | 2.028 | 2.434 | 2.719 | 3.333 | 3.582 |
| **38** | | 1.686 | 2.024 | 2.429 | 2.712 | 3.319 | 3.566 |
| **40** | | 1.684 | 2.021 | 2.423 | 2.704 | 3.307 | 3.551 |
| **42** | | 1.682 | 2.018 | 2.418 | 2.698 | 3.296 | 3.538 |
| **44** | | 1.680 | 2.015 | 2.414 | 2.692 | 3.286 | 3.526 |
| **46** | | 1.679 | 2.013 | 2.410 | 2.687 | 3.277 | 3.515 |
| **48** | | 1.677 | 2.011 | 2.407 | 2.682 | 3.269 | 3.505 |
| **50** | | 1.676 | 2.009 | 2.403 | 2.678 | 3.261 | 3.496 |
| **60** | | 1.671 | 2.000 | 2.390 | 2.660 | 3.232 | 3.460 |
| **70** | | 1.667 | 1.994 | 2.381 | 2.648 | 3.211 | 3.435 |
| **80** | | 1.664 | 1.990 | 2.374 | 2.639 | 3.195 | 3.416 |
| **90** | | 1.662 | 1.987 | 2.368 | 2.632 | 3.183 | 3.402 |
| **100** | | 1.660 | 1.984 | 2.364 | 2.626 | 3.174 | 3.390 |
| **120** | | 1.658 | 1.980 | 2.358 | 2.617 | 3.160 | 3.373 |
| **150** | | 1.655 | 1.976 | 2.351 | 2.609 | 3.145 | 3.357 |
| **200** | | 1.653 | 1.972 | 2.345 | 2.601 | 3.131 | 3.340 |
| **300** | | 1.650 | 1.968 | 2.339 | 2.592 | 3.118 | 3.323 |
| **400** | | 1.649 | 1.966 | 2.336 | 2.588 | 3.111 | 3.315 |
| **500** | | 1.648 | 1.965 | 2.334 | 2.586 | 3.107 | 3.310 |
| **600** | | 1.647 | 1.964 | 2.333 | 2.584 | 3.104 | 3.307 |
| $\infty$ | | 1.645 | 1.960 | 2.326 | 2.576 | 3.090 | 3.291 |

## 3 $F$ Distribution: Critical Values of $F$ (5% significance level)

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 161.45 | 199.50 | 215.71 | 224.58 | 230.16 | 233.99 | 236.77 | 238.88 | 240.54 | 241.88 | 243.91 | 245.36 | 246.46 | 247.32 | 248.01 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 | 19.40 | 19.41 | 19.42 | 19.43 | 19.44 | 19.45 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.71 | 8.69 | 8.67 | 8.66 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.87 | 5.84 | 5.82 | 5.80 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.64 | 4.60 | 4.58 | 4.56 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.96 | 3.92 | 3.90 | 3.87 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.53 | 3.49 | 3.47 | 3.44 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.24 | 3.20 | 3.17 | 3.15 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 3.03 | 2.99 | 2.96 | 2.94 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.86 | 2.83 | 2.80 | 2.77 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 | 2.85 | 2.79 | 2.74 | 2.70 | 2.67 | 2.65 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.64 | 2.60 | 2.57 | 2.54 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 | 2.67 | 2.60 | 2.55 | 2.51 | 2.48 | 2.46 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 | 2.60 | 2.53 | 2.48 | 2.44 | 2.41 | 2.39 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.42 | 2.38 | 2.35 | 2.33 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 | 2.42 | 2.37 | 2.33 | 2.30 | 2.28 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 | 2.45 | 2.38 | 2.33 | 2.29 | 2.26 | 2.23 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 | 2.34 | 2.29 | 2.25 | 2.22 | 2.19 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 | 2.38 | 2.31 | 2.26 | 2.21 | 2.18 | 2.16 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.28 | 2.22 | 2.18 | 2.15 | 2.12 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 | 2.25 | 2.20 | 2.16 | 2.12 | 2.10 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 | 2.30 | 2.23 | 2.17 | 2.13 | 2.10 | 2.07 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 | 2.27 | 2.20 | 2.15 | 2.11 | 2.08 | 2.05 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.18 | 2.13 | 2.09 | 2.05 | 2.03 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 | 2.24 | 2.16 | 2.11 | 2.07 | 2.04 | 2.01 |
| 26 | 4.22 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 | 2.22 | 2.15 | 2.09 | 2.05 | 2.02 | 1.99 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 | 2.20 | 2.13 | 2.08 | 2.04 | 2.00 | 1.97 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 | 2.19 | 2.12 | 2.06 | 2.02 | 1.99 | 1.96 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 | 2.18 | 2.10 | 2.05 | 2.01 | 1.97 | 1.94 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 | 2.16 | 2.09 | 2.04 | 1.99 | 1.96 | 1.93 |
| 35 | 4.12 | 3.27 | 2.87 | 2.64 | 2.49 | 2.37 | 2.29 | 2.22 | 2.16 | 2.11 | 2.04 | 1.99 | 1.94 | 1.91 | 1.88 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 | 2.08 | 2.00 | 1.95 | 1.90 | 1.87 | 1.84 |
| 50 | 4.03 | 3.18 | 2.79 | 2.56 | 2.40 | 2.29 | 2.20 | 2.13 | 2.07 | 2.03 | 1.95 | 1.89 | 1.85 | 1.81 | 1.78 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 | 1.99 | 1.92 | 1.86 | 1.82 | 1.78 | 1.75 |
| 70 | 3.98 | 3.13 | 2.74 | 2.50 | 2.35 | 2.23 | 2.14 | 2.07 | 2.02 | 1.97 | 1.89 | 1.84 | 1.79 | 1.75 | 1.72 |
| 80 | 3.96 | 3.11 | 2.72 | 2.49 | 2.33 | 2.21 | 2.13 | 2.06 | 2.00 | 1.95 | 1.88 | 1.82 | 1.77 | 1.73 | 1.70 |
| 90 | 3.95 | 3.10 | 2.71 | 2.47 | 2.32 | 2.20 | 2.11 | 2.04 | 1.99 | 1.94 | 1.86 | 1.80 | 1.76 | 1.72 | 1.69 |
| 100 | 3.94 | 3.09 | 2.70 | 2.46 | 2.31 | 2.19 | 2.10 | 2.03 | 1.97 | 1.93 | 1.85 | 1.79 | 1.75 | 1.71 | 1.68 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 | 1.91 | 1.83 | 1.78 | 1.73 | 1.69 | 1.66 |
| 150 | 3.90 | 3.06 | 2.66 | 2.43 | 2.27 | 2.16 | 2.07 | 2.00 | 1.94 | 1.89 | 1.82 | 1.76 | 1.71 | 1.67 | 1.64 |
| 200 | 3.89 | 3.04 | 2.65 | 2.42 | 2.26 | 2.14 | 2.06 | 1.98 | 1.93 | 1.88 | 1.80 | 1.74 | 1.69 | 1.66 | 1.62 |
| 250 | 3.88 | 3.03 | 2.64 | 2.41 | 2.25 | 2.13 | 2.05 | 1.98 | 1.92 | 1.87 | 1.79 | 1.73 | 1.68 | 1.65 | 1.61 |
| 300 | 3.87 | 3.03 | 2.63 | 2.40 | 2.24 | 2.13 | 2.04 | 1.97 | 1.91 | 1.86 | 1.78 | 1.72 | 1.68 | 1.64 | 1.61 |
| 400 | 3.86 | 3.02 | 2.63 | 2.39 | 2.24 | 2.12 | 2.03 | 1.96 | 1.90 | 1.85 | 1.78 | 1.72 | 1.67 | 1.63 | 1.60 |
| 500 | 3.86 | 3.01 | 2.62 | 2.39 | 2.23 | 2.12 | 2.03 | 1.96 | 1.90 | 1.85 | 1.77 | 1.71 | 1.66 | 1.62 | 1.59 |
| 600 | 3.86 | 3.01 | 2.62 | 2.39 | 2.23 | 2.11 | 2.02 | 1.95 | 1.90 | 1.85 | 1.77 | 1.71 | 1.66 | 1.62 | 1.59 |
| 750 | 3.85 | 3.01 | 2.62 | 2.38 | 2.23 | 2.11 | 2.02 | 1.95 | 1.89 | 1.84 | 1.77 | 1.70 | 1.66 | 1.62 | 1.58 |
| 1000 | 3.85 | 3.00 | 2.61 | 2.38 | 2.22 | 2.11 | 2.02 | 1.95 | 1.89 | 1.84 | 1.76 | 1.70 | 1.65 | 1.61 | 1.58 |

## $F$ Distribution: Critical Values of $F$ (5% significance level) continued

| $\nu_2$ \ $\nu_1$ | 25 | 30 | 35 | 40 | 50 | 60 | 75 | 100 | 150 | 200 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 249.26 | 250.10 | 250.69 | 251.14 | 251.77 | 252.20 | 252.62 | 253.04 | 253.46 | 253.68 |
| 2 | 19.46 | 19.46 | 19.47 | 19.47 | 19.48 | 19.48 | 19.48 | 19.49 | 19.49 | 19.49 |
| 3 | 8.63 | 8.62 | 8.60 | 8.59 | 8.58 | 8.57 | 8.56 | 8.55 | 8.54 | 8.54 |
| 4 | 5.77 | 5.75 | 5.73 | 5.72 | 5.70 | 5.69 | 5.68 | 5.66 | 5.65 | 5.65 |
| 5 | 4.52 | 4.50 | 4.48 | 4.46 | 4.44 | 4.43 | 4.42 | 4.41 | 4.39 | 4.39 |
| 6 | 3.83 | 3.81 | 3.79 | 3.77 | 3.75 | 3.74 | 3.73 | 3.71 | 3.70 | 3.69 |
| 7 | 3.40 | 3.38 | 3.36 | 3.34 | 3.32 | 3.30 | 3.29 | 3.27 | 3.26 | 3.25 |
| 8 | 3.11 | 3.08 | 3.06 | 3.04 | 3.02 | 3.01 | 2.99 | 2.97 | 2.96 | 2.95 |
| 9 | 2.89 | 2.86 | 2.84 | 2.83 | 2.80 | 2.79 | 2.77 | 2.76 | 2.74 | 2.73 |
| 10 | 2.73 | 2.70 | 2.68 | 2.66 | 2.64 | 2.62 | 2.60 | 2.59 | 2.57 | 2.56 |
| 11 | 2.60 | 2.57 | 2.55 | 2.53 | 2.51 | 2.49 | 2.47 | 2.46 | 2.44 | 2.43 |
| 12 | 2.50 | 2.47 | 2.44 | 2.43 | 2.40 | 2.38 | 2.37 | 2.35 | 2.33 | 2.32 |
| 13 | 2.41 | 2.38 | 2.36 | 2.34 | 2.31 | 2.30 | 2.28 | 2.26 | 2.24 | 2.23 |
| 14 | 2.34 | 2.31 | 2.28 | 2.27 | 2.24 | 2.22 | 2.21 | 2.19 | 2.17 | 2.16 |
| 15 | 2.28 | 2.25 | 2.22 | 2.20 | 2.18 | 2.16 | 2.14 | 2.12 | 2.10 | 2.10 |
| 16 | 2.23 | 2.19 | 2.17 | 2.15 | 2.12 | 2.11 | 2.09 | 2.07 | 2.05 | 2.04 |
| 17 | 2.18 | 2.15 | 2.12 | 2.10 | 2.08 | 2.06 | 2.04 | 2.02 | 2.00 | 1.99 |
| 18 | 2.14 | 2.11 | 2.08 | 2.06 | 2.04 | 2.02 | 2.00 | 1.98 | 1.96 | 1.95 |
| 19 | 2.11 | 2.07 | 2.05 | 2.03 | 2.00 | 1.98 | 1.96 | 1.94 | 1.92 | 1.91 |
| 20 | 2.07 | 2.04 | 2.01 | 1.99 | 1.97 | 1.95 | 1.93 | 1.91 | 1.89 | 1.88 |
| 21 | 2.05 | 2.01 | 1.98 | 1.96 | 1.94 | 1.92 | 1.90 | 1.88 | 1.86 | 1.84 |
| 22 | 2.02 | 1.98 | 1.96 | 1.94 | 1.91 | 1.89 | 1.87 | 1.85 | 1.83 | 1.82 |
| 23 | 2.00 | 1.96 | 1.93 | 1.91 | 1.88 | 1.86 | 1.84 | 1.82 | 1.80 | 1.79 |
| 24 | 1.97 | 1.94 | 1.91 | 1.89 | 1.86 | 1.84 | 1.82 | 1.80 | 1.78 | 1.77 |
| 25 | 1.96 | 1.92 | 1.89 | 1.87 | 1.84 | 1.82 | 1.80 | 1.78 | 1.76 | 1.75 |
| 26 | 1.94 | 1.90 | 1.87 | 1.85 | 1.82 | 1.80 | 1.78 | 1.76 | 1.74 | 1.73 |
| 27 | 1.92 | 1.88 | 1.86 | 1.84 | 1.81 | 1.79 | 1.76 | 1.74 | 1.72 | 1.71 |
| 28 | 1.91 | 1.87 | 1.84 | 1.82 | 1.79 | 1.77 | 1.75 | 1.73 | 1.70 | 1.69 |
| 29 | 1.89 | 1.85 | 1.83 | 1.81 | 1.77 | 1.75 | 1.73 | 1.71 | 1.69 | 1.67 |
| 30 | 1.88 | 1.84 | 1.81 | 1.79 | 1.76 | 1.74 | 1.72 | 1.70 | 1.67 | 1.66 |
| 35 | 1.82 | 1.79 | 1.76 | 1.74 | 1.70 | 1.68 | 1.66 | 1.63 | 1.61 | 1.60 |
| 40 | 1.78 | 1.74 | 1.72 | 1.69 | 1.66 | 1.64 | 1.61 | 1.59 | 1.56 | 1.55 |
| 50 | 1.73 | 1.69 | 1.66 | 1.63 | 1.60 | 1.58 | 1.55 | 1.52 | 1.50 | 1.48 |
| 60 | 1.69 | 1.65 | 1.62 | 1.59 | 1.56 | 1.53 | 1.51 | 1.48 | 1.45 | 1.44 |
| 70 | 1.66 | 1.62 | 1.59 | 1.57 | 1.53 | 1.50 | 1.48 | 1.45 | 1.42 | 1.40 |
| 80 | 1.64 | 1.60 | 1.57 | 1.54 | 1.51 | 1.48 | 1.45 | 1.43 | 1.39 | 1.38 |
| 90 | 1.63 | 1.59 | 1.55 | 1.53 | 1.49 | 1.46 | 1.44 | 1.41 | 1.38 | 1.36 |
| 100 | 1.62 | 1.57 | 1.54 | 1.52 | 1.48 | 1.45 | 1.42 | 1.39 | 1.36 | 1.34 |
| 120 | 1.60 | 1.55 | 1.52 | 1.50 | 1.46 | 1.43 | 1.40 | 1.37 | 1.33 | 1.32 |
| 150 | 1.58 | 1.54 | 1.50 | 1.48 | 1.44 | 1.41 | 1.38 | 1.34 | 1.31 | 1.29 |
| 200 | 1.56 | 1.52 | 1.48 | 1.46 | 1.41 | 1.39 | 1.35 | 1.32 | 1.28 | 1.26 |
| 250 | 1.55 | 1.50 | 1.47 | 1.44 | 1.40 | 1.37 | 1.34 | 1.31 | 1.27 | 1.25 |
| 300 | 1.54 | 1.50 | 1.46 | 1.43 | 1.39 | 1.36 | 1.33 | 1.30 | 1.26 | 1.23 |
| 400 | 1.53 | 1.49 | 1.45 | 1.42 | 1.38 | 1.35 | 1.32 | 1.28 | 1.24 | 1.22 |
| 500 | 1.53 | 1.48 | 1.45 | 1.42 | 1.38 | 1.35 | 1.31 | 1.28 | 1.23 | 1.21 |
| 600 | 1.52 | 1.48 | 1.44 | 1.41 | 1.37 | 1.34 | 1.31 | 1.27 | 1.23 | 1.20 |
| 750 | 1.52 | 1.47 | 1.44 | 1.41 | 1.37 | 1.34 | 1.30 | 1.26 | 1.22 | 1.20 |
| 1000 | 1.52 | 1.47 | 1.43 | 1.41 | 1.36 | 1.33 | 1.30 | 1.26 | 1.22 | 1.19 |

## 4 $\chi^2$ (Chi-Squared) Distribution: Critical Values of $\chi^2$

Values $\chi^2_{k,p}$ of $x$ for which $\mathbb{P}(X > x) = p$, when $X$ is $\chi^2_k$ and $p = 0.995$, $p = 0.975$, etc.

| $k$ | .995 | .975 | .05 | .025 | .01 | .005 | $k$ | .995 | .975 | .05 | .025 | .01 | .005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .000 | .001 | 3.84 | 5.02 | 6.63 | 7.88 | 18 | 6.26 | 8.23 | 28.87 | 31.53 | 34.81 | 37.16 |
| 2 | .010 | .051 | 5.99 | 7.38 | 9.21 | 10.60 | 20 | 7.43 | 9.59 | 31.42 | 34.17 | 37.57 | 40.00 |
| 3 | .072 | .216 | 7.81 | 9.35 | 11.34 | 12.84 | 22 | 8.64 | 10.98 | 33.92 | 36.78 | 40.29 | 42.80 |
| 4 | .207 | .484 | 9.49 | 11.14 | 13.28 | 14.86 | 24 | 9.89 | 12.40 | 36.42 | 39.36 | 42.98 | 45.56 |
| 5 | .412 | .831 | 11.07 | 12.83 | 15.09 | 16.75 | 26 | 11.16 | 13.84 | 38.89 | 41.92 | 45.64 | 48.29 |
| 6 | .676 | 1.24 | 12.59 | 14.45 | 16.81 | 18.55 | 28 | 12.46 | 15.31 | 41.34 | 44.46 | 48.28 | 50.99 |
| 7 | .990 | 1.69 | 14.07 | 16.01 | 18.48 | 20.28 | 30 | 13.79 | 16.79 | 43.77 | 46.98 | 50.89 | 53.67 |
| 8 | 1.34 | 2.18 | 15.51 | 17.53 | 20.09 | 21.95 | 40 | 20.71 | 24.43 | 55.76 | 59.34 | 63.69 | 66.77 |
| 9 | 1.73 | 2.70 | 16.92 | 19.02 | 21.67 | 23.59 | 50 | 27.99 | 32.36 | 67.50 | 71.41 | 76.15 | 79.49 |
| 10 | 2.16 | 3.25 | 13.31 | 20.48 | 23.21 | 25.19 | 60 | 35.53 | 40.48 | 79.08 | 83.30 | 88.38 | 91.95 |
| 12 | 3.07 | 4.40 | 21.03 | 23.34 | 26.22 | 28.30 | 70 | 43.28 | 48.76 | 90.53 | 95.02 | 100.4 | 104.2 |
| 14 | 4.07 | 5.63 | 23.68 | 26.12 | 29.14 | 31.32 | 80 | 51.17 | 57.15 | 101.9 | 106.6 | 112.3 | 116.3 |
| 16 | 5.14 | 6.91 | 26.30 | 28.85 | 32.00 | 34.27 | 100 | 67.33 | 74.22 | 124.3 | 129.6 | 135.8 | 140.2 |

## 5 Wilcoxon Rank Sum Test Critical Values (2.5%)

| $n_2$ \ $n_1$ | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $T_L$ | $T_U$ | $T_L$ | $T_U$ | $T_L$ | $T_U$ | $T_L$ | $T_U$ | $T_L$ | $T_U$ | $T_L$ | $T_U$ | $T_L$ | $T_U$ | $T_L$ | $T_U$ |
| 3 | 5 | 16 | 6 | 18 | 6 | 21 | 7 | 23 | 7 | 26 | 8 | 28 | 8 | 31 | 9 | 33 |
| 4 | 6 | 18 | 11 | 25 | 12 | 28 | 12 | 32 | 13 | 35 | 14 | 38 | 15 | 41 | 16 | 44 |
| 5 | 6 | 21 | 12 | 28 | 18 | 37 | 19 | 41 | 20 | 45 | 21 | 49 | 22 | 53 | 24 | 56 |
| 6 | 7 | 23 | 12 | 32 | 19 | 41 | 26 | 52 | 28 | 56 | 29 | 61 | 31 | 65 | 32 | 70 |
| 7 | 7 | 26 | 13 | 35 | 20 | 45 | 28 | 56 | 37 | 68 | 39 | 73 | 41 | 78 | 43 | 83 |
| 8 | 8 | 28 | 14 | 38 | 21 | 49 | 29 | 61 | 39 | 73 | 49 | 87 | 51 | 93 | 54 | 98 |
| 9 | 8 | 31 | 15 | 41 | 22 | 53 | 31 | 65 | 41 | 78 | 51 | 93 | 63 | 108 | 66 | 114 |
| 10 | 9 | 33 | 16 | 44 | 24 | 56 | 32 | 70 | 43 | 83 | 54 | 98 | 66 | 114 | 79 | 131 |

## 6 Wilcoxon Signed Rank Test Critical Values (2.5%)

| One-Tailed | Two-Tailed | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n = 9$ | $n = 10$ |
|---|---|---|---|---|---|---|---|
| $\alpha = .05$ | $\alpha = .10$ | 1 | 2 | 4 | 6 | 8 | 11 |
| $\alpha = .025$ | $\alpha = .05$ | | 1 | 2 | 4 | 6 | 8 |
| $\alpha = .01$ | $\alpha = .02$ | | | 0 | 2 | 3 | 5 |
| $\alpha = .005$ | $\alpha = .01$ | | | | 0 | 2 | 3 |
| | | $n = 11$ | $n = 12$ | $n = 13$ | $n = 14$ | $n = 15$ | $n = 16$ |
| $\alpha = .05$ | $\alpha = .10$ | 14 | 17 | 21 | 26 | 30 | 36 |
| $\alpha = .025$ | $\alpha = .05$ | 11 | 14 | 17 | 21 | 25 | 30 |
| $\alpha = .01$ | $\alpha = .02$ | 7 | 10 | 13 | 16 | 20 | 24 |
| $\alpha = .005$ | $\alpha = .01$ | 5 | 7 | 10 | 13 | 16 | 19 |
| | | $n = 17$ | $n = 18$ | $n = 19$ | $n = 20$ | $n = 21$ | $n = 22$ |
| $\alpha = .05$ | $\alpha = .10$ | 41 | 47 | 54 | 60 | 68 | 75 |
| $\alpha = .025$ | $\alpha = .05$ | 35 | 40 | 46 | 52 | 59 | 66 |
| $\alpha = .01$ | $\alpha = .02$ | 28 | 33 | 38 | 43 | 49 | 56 |
| $\alpha = .005$ | $\alpha = .01$ | 23 | 28 | 32 | 37 | 43 | 49 |
| | | $n = 23$ | $n = 24$ | $n = 25$ | $n = 26$ | $n = 27$ | $n = 28$ |
| $\alpha = .05$ | $\alpha = .10$ | 83 | 92 | 101 | 110 | 120 | 130 |
| $\alpha = .025$ | $\alpha = .05$ | 73 | 81 | 90 | 98 | 107 | 117 |
| $\alpha = .01$ | $\alpha = .02$ | 62 | 69 | 77 | 85 | 93 | 102 |
| $\alpha = .005$ | $\alpha = .01$ | 55 | 61 | 68 | 76 | 84 | 92 |

| | | $n = 29$ | $n = 30$ | $n = 31$ | $n = 32$ | $n = 33$ | $n = 34$ |
|---|---|---|---|---|---|---|---|
| $\alpha = .05$ | $\alpha = .10$ | 141 | 152 | 163 | 175 | 188 | 201 |
| $\alpha = .025$ | $\alpha = .05$ | 127 | 137 | 148 | 159 | 171 | 183 |
| $\alpha = .01$ | $\alpha = .02$ | 111 | 120 | 130 | 141 | 151 | 162 |
| $\alpha = .005$ | $\alpha = .01$ | 100 | 109 | 118 | 128 | 138 | 149 |
| | | $n = 35$ | $n = 36$ | $n = 37$ | $n = 38$ | $n = 39$ | |
| $\alpha = .05$ | $\alpha = .10$ | 214 | 228 | 242 | 256 | 271 | |
| $\alpha = .025$ | $\alpha = .05$ | 195 | 208 | 222 | 235 | 250 | |
| $\alpha = .01$ | $\alpha = .02$ | 174 | 186 | 198 | 211 | 224 | |
| $\alpha = .005$ | $\alpha = .01$ | 160 | 171 | 183 | 195 | 208 | |
| | | $n = 40$ | $n = 41$ | $n = 42$ | $n = 43$ | $n = 44$ | $n = 45$ |
| $\alpha = .05$ | $\alpha = .10$ | 287 | 303 | 319 | 336 | 353 | 371 |
| $\alpha = .025$ | $\alpha = .05$ | 264 | 279 | 295 | 311 | 327 | 344 |
| $\alpha = .01$ | $\alpha = .02$ | 238 | 252 | 267 | 281 | 297 | 313 |
| $\alpha = .005$ | $\alpha = .01$ | 221 | 234 | 248 | 262 | 277 | 292 |
| | | $n = 46$ | $n = 47$ | $n = 48$ | $n = 49$ | $n = 50$ | |
| $\alpha = .05$ | $\alpha = .10$ | 389 | 408 | 427 | 446 | 466 | |
| $\alpha = .025$ | $\alpha = .05$ | 361 | 379 | 397 | 415 | 434 | |
| $\alpha = .01$ | $\alpha = .02$ | 329 | 345 | 362 | 380 | 398 | |
| $\alpha = .005$ | $\alpha = .01$ | 307 | 323 | 339 | 356 | 373 | |