## The Fourier Series & Transforms

| Equation | Discrete Time | Continuous Time |
|---|---|---|
| Fourier Series (Analysis) | $X_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x(n)e^{-ik\omega_0 n}$ | $X_k = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-ik\omega_0 t}dt$ |
| Inverse Fourier Series (Synthesis) | $x(n) = \sum_{k\in\langle N\rangle} X_k e^{ik\omega_0 n}$ | $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$ |
| DFT (Analysis) | $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-ik\omega_0 n}$ | N/A |
| Inverse DFT (Synthesis) | $x(n) = \sum_{k=0}^{N-1} X[k]e^{ik\omega_0 n}$ | N/A |
| Fourier Transform (Analysis) | $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$ | $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$ |
| Inverse Fourier Transform (Synthesis) | $x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{i\omega n}d\omega$ | $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$ |
| Z-Transform | $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$ | N/A |

**Note**: $\langle p\rangle$ represents a period of $p$ (e.g. $(-\frac{1}{2}p, \frac{1}{2}p)$ or $(0,p)$). For the equations, $\omega \leftrightarrow 2\pi f$, $\omega_0 \leftrightarrow 2\pi f_0$, and $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ or $N = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$.

## Fourier Series Properties

| Property | DTFS | CTFS |
|---|---|---|
| Linearity | $\alpha x(n) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X_k$<br>$x(n)+y(n) \overset{\mathcal{F}}{\longleftrightarrow} X_k + Y_k$ | $\alpha x(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X_k$<br>$x(t)+y(t) \overset{\mathcal{F}}{\longleftrightarrow} X_k + Y_k$ |
| Convolution | $(x*y)(n) \overset{\mathcal{F}}{\longleftrightarrow} X_k \cdot Y_k$ | $(x*y)(t) \overset{\mathcal{F}}{\longleftrightarrow} X_k \cdot Y_k$ |
| Time Reversal | $x(-n) \overset{\mathcal{F}}{\longleftrightarrow} X_{-k}$ | $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X_{-k}$ |
| Conjugation | $x^*(n) \overset{\mathcal{F}}{\longleftrightarrow} X^*_{-k}$ | $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*_{-k}$ |
| Conjugate Symmetry | $x(n)\in\mathbb{R} \iff X_k = X^*_{-k}$ | $x(t)\in\mathbb{R} \iff X_k = X^*_{-k}$ |
| Real and Even $x$ | Real and even $X_k$ | |
| Real and Odd $x$ | Imaginary and odd $X_k$ | |
| Imaginary and Even $x$ | Imaginary and even $X_k$ | |
| Imaginary and Odd $x$ | Real and odd $X_k$ | |
| Time Shift | $x(n-n_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-ik\omega_0 n_0}X_k$ | $x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-ik\omega_0 t_0}X_k$ |
| Parseval's | $\frac{1}{N}\sum\limits_{n\in\langle N\rangle} \lvert x(n)\rvert^2 = \sum\limits_{k\in\langle N\rangle} \lvert X_k\rvert^2$ | $\frac{1}{T}\int_{\langle T\rangle} \lvert x(t)\rvert^2 dt = \sum\limits_{k=-\infty}^{\infty} \lvert X_k\rvert^2$ |
| Poisson's | $\sum\limits_{l=-\infty}^{\infty} \delta(n-lN) = \frac{1}{N}\sum\limits_{k\in\langle N\rangle} e^{ik\omega_0 n}$ | $\sum\limits_{l=-\infty}^{\infty} \delta(t-lT) = \frac{1}{T}\sum\limits_{k=-\infty}^{\infty} e^{ik\omega_0 t}$ |

**Fourier Transform Pairs**

| Pairs | Discrete Time | Continuous Time |
|---|---|---|
| Constant | $1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ | $1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega)$ |
| Complex Exponential | $e^{i\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$ | $e^{i\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$ |
| Exponential | $\alpha^n u(n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-i\omega}},\ \lvert\alpha\rvert < 1$ | $e^{-\alpha t} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{i\omega+\alpha}, \mathfrak{Re}(\alpha) > 0$ |
| Delta | $\delta(n) \overset{\mathcal{F}}{\longleftrightarrow} 1$ | $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$ |
| Shifted Delta | $\delta(n - n_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-i\omega n_0}$ | $\delta(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-i\omega t_0}$ |
| Rect in Time | $x(n) = \begin{cases} 1 & n \in \{-M, \ldots, M\} \\ 0 & \text{otherwise} \end{cases}$ <br> $X(\omega) = \begin{cases} \dfrac{\sin((\frac{2M+1}{2})\omega)}{\sin(\omega/2)} & \omega \neq 0 \\ 2M+1 & \omega = 0 \end{cases}$ | $x(t) = \begin{cases} 1 & \lvert t\rvert \leq B \\ 0 & \lvert t\rvert > B \end{cases}$ <br> $X(\omega) = 2B \operatorname{sinc}\left(\frac{B}{\pi}\omega\right)$ |
| Rect in Freq. | $f(n) = \frac{\omega_0}{\pi} \operatorname{sinc}\left(\frac{\omega_0}{\pi} n\right)$ <br> $F(\omega) = \begin{cases} 1 & \omega \in [-\omega_0 + 2\pi k, \omega_0 + 2\pi k] \\ 0 & \text{otherwise} \end{cases}$ | $f(t) = \frac{\omega_0}{\pi} \operatorname{sinc}\left(\frac{\omega_0}{\pi} t\right)$ <br> $F(\omega) = \begin{cases} 1 & \omega \in [-\omega_0, \omega_0] \\ 0 & \text{otherwise} \end{cases}$ |
| Delta Train | $s(n) = \sum_{l=-\infty}^{\infty} \delta(n - lN)$ <br> $S(\omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{N})$ | $s(t) = \sum_{l=-\infty}^{\infty} \delta(t - lT)$ <br> $S(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$ |

**Note**: $\omega = 2\pi f$.

**Important Equations**

| Name | Equation |
|---|---|
| Sinc | $\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ |
| Euler's | $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ |
| Inverse Euler's | $\cos(\omega) = \frac{1}{2}\left(e^{i\omega} + e^{-i\omega}\right)$ <br> $\sin(\omega) = \frac{1}{2i}\left(e^{i\omega} - e^{-i\omega}\right)$ |
| Infinite Geometric Series Summation | $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad \lvert r\rvert < 1$ |
| Finite Geometric Series Summation | $\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1-r} \quad r \neq 1$ |

**Fourier Transform Properties**

| Property | DTFT | CTFT |
|---|---|---|
| Linearity | $\alpha x(n) \xleftrightarrow{\mathcal{F}} \alpha X(\omega)$<br>$x(n)+y(n) \xleftrightarrow{\mathcal{F}} X(\omega)+Y(\omega)$ | $\alpha x(t) \xleftrightarrow{\mathcal{F}} \alpha X(\omega)$<br>$x(t)+y(t) \xleftrightarrow{\mathcal{F}} X(\omega)+Y(\omega)$ |
| Convolution | $(x * y)(n) \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$ | $(x * y)(t) \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$ |
| Modulation | $x(n)y(n) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}(X \circledast Y)(\omega)$ | $x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}(X * Y)(\omega)$ |
| Downsampling | $x(an) \xleftrightarrow{\mathcal{F}} \frac{1}{a}\sum_{k=0}^{a-1} X\left(\frac{\omega-2k\pi}{a}\right),\ a \in \mathbb{Z}^+$ | $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$ |
| Upsampling | $\begin{cases} x\left(\frac{n}{a}\right) & a \mid n \\ 0 & a \nmid n \end{cases} \xleftrightarrow{\mathcal{F}} X(a\omega),\ a \in \mathbb{Z}^+$ | $x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} \lvert a\rvert X(a\omega)$ |
| Time-Reversal | $x(-n) \xleftrightarrow{\mathcal{F}} X(-\omega)$ | $x(-t) \xleftrightarrow{\mathcal{F}} X(-\omega)$ |
| Conjugation | $x^*(n) \xleftrightarrow{\mathcal{F}} X^*(-\omega)$ | $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-\omega)$ |
| Conjugate Symmetry | $x(n) \in \mathbb{R} \iff X(\omega) = X^*(-\omega)$ | $x(t) \in \mathbb{R} \iff X(\omega) = X^*(-\omega)$ |
| Real and Even $x$ | Real and even $X(\omega)$ | |
| Real and Odd $x$ | Imaginary and odd $X(\omega)$ | |
| Imaginary and Even $x$ | Imaginary and even $X(\omega)$ | |
| Imaginary and Odd $x$ | Real and odd $X(\omega)$ | |
| Time Shift | $x(n-n_0) \xleftrightarrow{\mathcal{F}} e^{-i\omega n_0} X(\omega)$ | $x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-i\omega t_0} X(\omega)$ |
| Frequency Shift | $e^{i\omega_0 n} x(n) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$ | $e^{i\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$ |
| Time Differentiation | N/A | $\frac{d}{dt} x(t) \xleftrightarrow{\mathcal{F}} i\omega X(\omega)$ |
| Freq. Differentiation | $n x(n) \xleftrightarrow{\mathcal{F}} i\frac{d}{d\omega} X(\omega)$ | $t x(t) \xleftrightarrow{\mathcal{F}} i\frac{d}{d\omega} X(\omega)$ |
| Accumulation/Integration | $\sum_{k=-\infty}^{n} x(n) \xleftrightarrow{\mathcal{F}} \frac{X(\omega)}{1-e^{-i\omega}} + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ | $\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{X(\omega)}{i\omega} + \pi X(0)\delta(\omega)$ |
| Parseval's | $\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} \lvert X(\omega)\rvert^2 d\omega$ | $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$ |
| Duality | $x(n) \xleftrightarrow{\text{DTFT}} X(\omega)$<br>$X(t) \xleftrightarrow{\text{CTFS } p=2\pi} X_k = x(-k)$ | $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$<br>$X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$ |

**Note**: $(X \circledast Y)(\omega)$ is the circular, or periodic convolution of $X(\omega)$ and $Y(\omega)$. This is equivalent to convolving $Y(\omega)$ with one $2\pi$ period of $X(\omega)$ (and the rest of $X(\omega)$ zeroed out), or vice versa. $a \mid n$ means $n$ is divisible by $a$, and $a \nmid n$ means $n$ is not divisible by $a$.

**Z-Transform Properties & Pairs**

| Property | Z-Transform | Region of Convergence |
|---|---|---|
| Linearity | $\alpha x(n) + \beta y(n) \overset{\mathcal{Z}}{\longleftrightarrow} \alpha X(z) + \beta Y(z)$ | $\supseteq \mathsf{RoC}(x) \cap \mathsf{RoC}(y)$ |
| Time Shift | $x(n - n_0) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z)$ | $\mathsf{RoC}(x)$ except $\begin{cases} z = 0 & n_0 > 0 \\ z = \infty & n_0 < 0 \end{cases}$ |
| $z$ Scaling | $a^n x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{a}\right)$ | $\|a\| \cdot \mathsf{RoC}(x)$ |
| Time Reversal | $x(-n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z^{-1})$ | $1/\mathsf{RoC}(x)$ |
| Convolution | $(x * h)(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)H(z)$ | $\supseteq \mathsf{RoC}(x) \cap \mathsf{RoC}(h)$ |
| Conjugation | $x^*(n) \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$ | $\mathsf{RoC}(x)$ |
| Conjugate Symmetry | $x(n) \in \mathbb{R} \overset{\mathcal{F}}{\longleftrightarrow} X(z) = X^*(z^*)$ | $\mathsf{RoC}(x)$ |
| $z$ Differentiation | $nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}$ | $\mathsf{RoC}(x)$ |
| Initial Value Theorem | $x(n) = 0, \forall n < 0 \overset{\mathcal{Z}}{\longleftrightarrow} \lim_{z\to\infty} X(z) = x(0)$ | Outside the outermost pole, out to, and including, $+\infty$ |

| Signal | Z-Transform | Region of Convergence |
|---|---|---|
| $\delta(n)$ | $1$ | $\{z \in \mathbb{C}\}$ |
| $\delta(n - n_0)$ | $z^{-n_0}$ | $\{z \in \mathbb{C}\}$ except $\begin{cases} z = 0 & n_0 > 0 \\ z = \infty & n_0 < 0 \end{cases}$ |
| $u(n)$ | $\frac{1}{1-z^{-1}}$ | $\{z \in \mathbb{C} \mid \|z\| > 1\}$ |
| $-u(-n-1)$ | $\frac{1}{1-z^{-1}}$ | $\{z \in \mathbb{C} \mid \|z\| < 1\}$ |
| $a^n u(n)$ | $\frac{1}{1-az^{-1}}$ | $\{z \in \mathbb{C} \mid \|z\| > \|a\|\}$ |
| $-a^n u(-n-1)$ | $\frac{1}{1-az^{-1}}$ | $\{z \in \mathbb{C} \mid \|z\| < \|a\|\}$ |
| $na^n u(n)$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\{z \in \mathbb{C} \mid \|z\| > \|a\|\}$ |
| $-na^n u(-n-1)$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\{z \in \mathbb{C} \mid \|z\| < \|a\|\}$ |
| $\cos(\omega_0 n)u(n)$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\{z \in \mathbb{C} \mid \|z\| > 1\}$ |
| $\sin(\omega_0 n)u(n)$ | $\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\{z \in \mathbb{C} \mid \|z\| > 1\}$ |
| $a^n \cos(\omega_0 n)u(n)$ | $\frac{1-a\cos(\omega_0)z^{-1}}{1-2a\cos(\omega_0)z^{-1}+a^2z^{-2}}$ | $\{z \in \mathbb{C} \mid \|z\| > \|a\|\}$ |
| $a^n \sin(\omega_0 n)u(n)$ | $\frac{a\sin(\omega_0)z^{-1}}{1-2a\cos(\omega_0)z^{-1}+a^2z^{-2}}$ | $\{z \in \mathbb{C} \mid \|z\| > \|a\|\}$ |