ME 103 Discussion 2

Week of 1/26

Probabilities and Normal Distribution

$n > 30 \rightarrow z$
$n < 30 \rightarrow t$

- General Concept: Taking your values, standardizing them to a normal distribution (from 0-1) and using mean(μ), std.dev(σ) and z table to find probabilities
- Z-score: n> 30, t-score n<30 UNLESS distribution is assumed to be normal
- Estimate of population mean = sample mean ($\bar{x}$) $\longrightarrow$ np.mean( )
- Estimate of population standard deviation = sample standard deviation (s) $\longrightarrow$ np.std(, ddof=1)
- Converting to z-score:
  - $Z = (X - \mu)/\sigma$ $\longleftarrow$
- Plug into z-table (first two digits on columns, last digit in first row), or norm.cdf in python
- Cumulative Density Function (CDF) value = percent of distribution below X. .
  - I.g. ϕ(1.96) = 95%--> 95% of the distribution exists at 1.96 and below.

$\mu = 6$

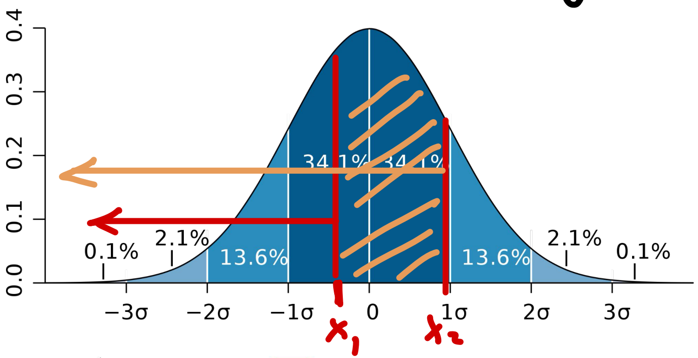
$CI = \pm 0.2$

$\alpha = 0.95$

$CI = \pm Z\sigma$

$0.2 = \pm 1.96\sigma$

$P(X > 6.050) = 1 - P(X < 6.050) = 1 - \Phi\left(\frac{x-\mu}{\sigma}\right)$



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414
-0.1	.46017	.45620	.45224	.44828	.44433	.44034	.43640	.43251	.42858	.42465
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-1	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08692	.08534	.08379	.08226
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-2	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-3	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-4	.00003	.00003	.00003	.00003	.00003	.00003	.00002	.00002	.00002	.00002

Basic Probability Rules!

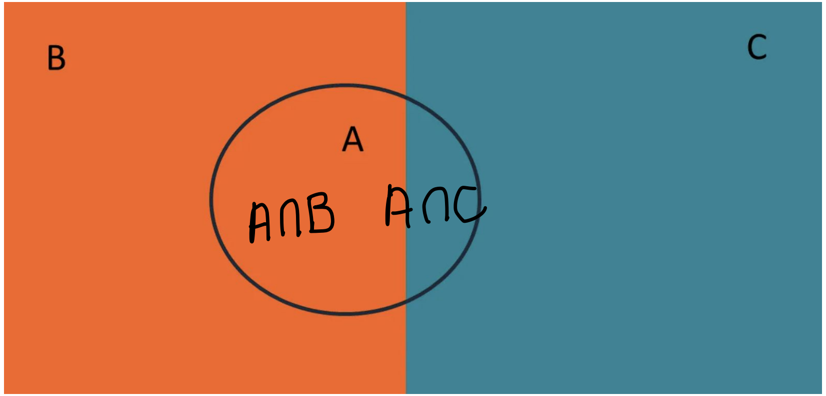
1. Possible values for probabilities range from 0 to 1
   a. 0 = impossible event, 1 = certain event
2. The sum of all the probabilities for all possible outcomes is equal to 1.
   a. $P(\Omega) = 1$, where Ω is the <u>probability space</u>- set of all possible events.
3. Complement Rule (Not A):
   a. $P(\bar{A}) = P(A^c) = 1 - P(A)$
4. Union Rule (A or B):
   intersection
   a. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. Product Rule (A and B):
   a. $P(A \cap B) = P(A) * P(B)$ ***for independent events
6. Conditional Probability (A *given* B):
   a. $P(A|B) = P(A \cap B) / P(B)$

Total Probability Rule

- By using conditional probabilities, we can find the probability of unknown event A.
- Ex. (right)
  - P(A) = P(A ∩ B) + P(A ∩ C)
- We also know: P(A ∩ B) = P(A|B) * P(B) (Conditional Rule)
- This can be summed through the nth conditional probability.

$$\mathbf{Pr}(B) = \mathbf{Pr}(A_1 \cap B) + \ldots + \mathbf{Pr}(A_n \cap B) = \sum_{i=1}^{n} \mathbf{Pr}(A_i)\,\mathbf{Pr}(B|A_i)$$

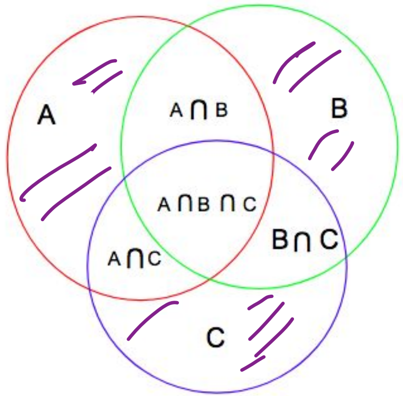
$$P(B) = \sum Pr(A)P(B|A) + P(C)P(B|C)$$



Inclusion/ Exclusion Principle

- Extension of **Union Law.**
- Math version →

$$\mathbf{Pr}\left(\bigcup_{i=1}^{n} A_i\right) = \sum \mathbf{Pr}(A_i) - \sum_{i<j} \mathbf{Pr}(A_i \cap A_j) + \sum_{i<j<k} \mathbf{Pr}(A_i \cap A_j \cap A_k) - \ldots$$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Bayes Theorem

- Rearrangement of previous rules

Conditional probability: Bayes' Theorem

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Probability Potpourri Part a I

- Split into random vars:
  - P(W): probability windy = 0.3
  - P(H) : probability hit
  - P(H|W) = 0.4 : “archer hits her target **given** it is windy
- Find probability it is windy and she hits the shot= P(H∩W)
  - Solve using probability rules and knowns
- Find the probability she hits on her first shot.
  - You have two scenarios, that she hits and it is windy + probability she hits and its not windy.
  - Whenever you have the probability of an event over multiple scenarios, use total probability rule.

$P(W) = 0.3 \quad P(\bar{W}) = 0.7 \quad P(H|\bar{W}) = 0.7$

$P(H) = \quad P(\bar{H}) =$

$P(H|W) = 0.4 \quad P(H \cap W) = \checkmark$

$P(\bar{H}|W) = 1 - P(H|W) = 0.6$

or equivalently,

$$\boxed{P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_k)P(B_k).}$$

^^ total probability rule

$B_1 = W \quad B_2 = \bar{W}$

$P(H) = P(H|W)P(W) + P(H|\bar{W})P(\bar{W})$

Probability Potpourri II

- Find the probability she hits **once in two shots:**
  - Number of trials: n =2
  - Number of successes: x=1
  - Probability of success (she hits once) = p calculated from previous question $P(H)$
- Use the binomial distribution formula

$P(\bar{H}) = 1 - P(H)$

Binomial Distribution Formula

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$$P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{(n-x)!x!} p^x q^{n-x}$$

where

n = the number of trials (or the number being sampled)

x = the number of successes desired

p = probability of getting a success in one trial

q = 1 - p = the probability of getting a failure in one trial

Probability Potpourri III

- "on an occasion when she missed, there was no gust of wind."
  - No wind, **given** no hit
  - You have every probability now, can use Bayes

$$P(\bar{W}|\bar{H}) = \frac{P(\bar{H}|\bar{W})P(\bar{W})}{P(\bar{H})}$$

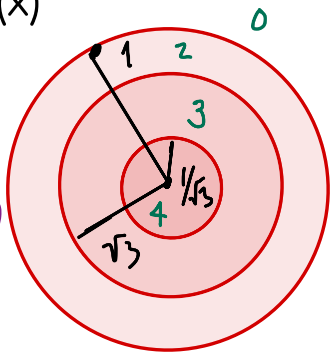
$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$f(x) = \frac{2}{\pi(1+x^2)}$ $\quad E[X] = \int_0^{1/\sqrt{3}} 4\,f(x)\,dx + \int_{1/\sqrt{3}}^{\sqrt{3}} 3\,f(x) + \dots$

Expected Value

- X = point values, 4, 3, 2
- Plug into matlab or solve by hand
- Separate into 3 limits (0,1/√3), (1/√3, √3), (√3,1)
- Helpful rule: int (1)/(1+x^2) = arctan(x)

$\int \frac{1}{1+x^2} = \arctan(x)$



$$E[X] = \sum_i x_i f(x_i)$$

discrete

μ

continuous

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

Shock Dyno

$X \to \{1, \ldots, 6\} \to$ discrete Uniform

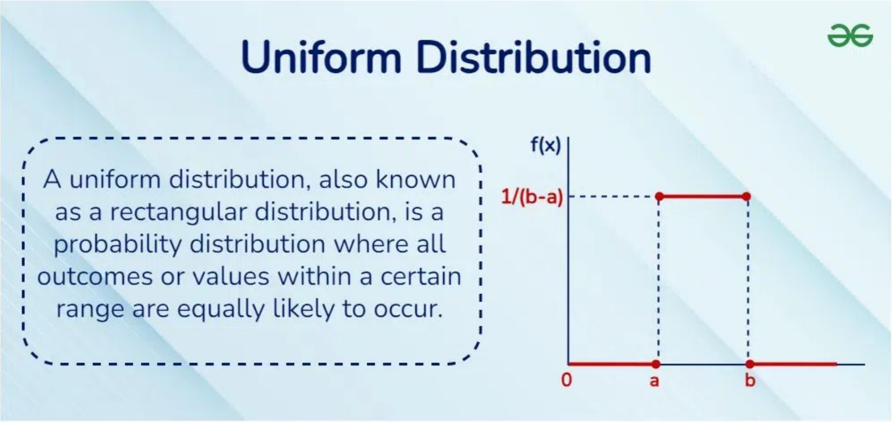
↑ 100x

$Y \to$ 1 or 0

$P(Y) = 0.5$

Binomial ← 600X

- 2 random variables, X and Y.
- You're looking for P(X<Y), or P(X-Y) < 0. Just like before, need to convert to z scores and find CDF
- Pick out "**equal likelihood per stroke**" gives you a hint that you are using uniform distribution for X.
- {1,...,6} hints that it is **a discrete uniform distribution.**
  - Find E[X]: n = 6,
  - Find var of X
- Since you have 100 independent samples, The total variance = the variance of 1 trial x 100.



Discrete Uniform: $X \sim \text{Uniform}(n)$

$$p_x = \frac{1}{n}\ (x = 1, 2, \ldots, n) \quad \mu = (n+1)/2, \ \sigma^2 = (n^2 - 1)/12$$

E[X]

Var[σ^2]

Shock Dyno

- For Y: “Each window independently records 1 (event) or 0 (no event) with probability 0.5 each. “
  - Whenever you have a chance of 1 or 0 only, success/fail, that’s a hint that you use Binomial distribution.
  - n= 600
  - p = 0.5
  - Find mean and variance, remember var * 600 trials = total variance.
- Define random variable Z = X - Y
- We can assume normal distribution.
- Find E[Z] = E[X-Y] and Var[Z] = Var[X - Y]. Cov = 0 for independent variables.
- Finally, convert 0 to z score and use CDF to find percent of distribution under (0 - E[Z])/√Var[Z]

Binomial distribution equations

$\text{Mean} = \mu = E(x) = np$ ← $E[Y]$

$\text{Variance} = \sigma^2 = np(1-p)$ ← $Var[Y]$

$\text{Standard Deviation} = \sigma = \sqrt{np(1-p)}$

where

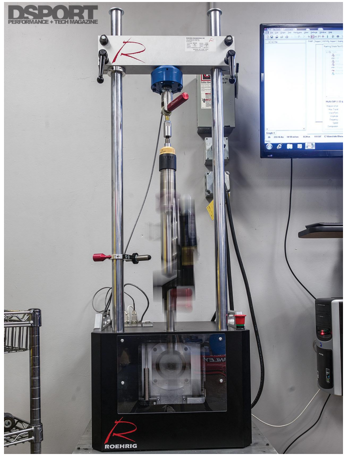
n = number of trials $E[X] = 100 \cdot E[X_i]$

p = probability of success $Var[X] = 100 \cdot Var[X_i]$

1-p = probability of other outcome (failure)

$\mathbb{E}[\cdot]$ is a linear operator: $E[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X, Y)$$



Larry Time

(d) [6 pts] You meet two students in Hesse Hall. Assume that each student is a Senior or a Sophomore with equal probability, and each student takes ME103 with probability $1/10$, independent of each other and independent of their class standing. What is the probability that both students are Seniors, given at least one of them is a Senior currently taking ME103?

(e) [6 pts] There are n identical looking sensors in Hesse 122 and n boxes to store them, where sensor i is supposed to go into box i. After a hectic cleanup, the sensors get thrown into the boxes uniformly at random (every permutation is equally likely). Assume n is a positive integer. What is the probability that **no sensor** ends up in its correct bin? Also find the limit of this probability as $n \to \infty$.

Hint: Use the principle of inclusion-exclusion and the power series representation of e^x. Assume that each box gets exactly one sensor.